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# STRESS-INTENSITY FACTORS FOR A SINGLE-EDGE-NOTCH TENSION SPECIMEN BY BOUNDARY COLLOCATION OF A STRESS FUNCTION

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# STRESS-INTENSITY FACTORS FOR A SINGLE-EDGE-NOTCH TENSION SPECIMEN BY BOUNDARY COLLOCATION OF A STRESS FUNCTION

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## SUMMARY

A boundary value collocation procedure applied to the Williams stress function was employed to determine the elastic stress distribution in the immediate vicinity of the tip of an edge crack in a finite-width specimen subjected to uniform tensile loading. This type of single-edge-notch specimen is particularly suitable for determination of plane strain fracture toughness values. The analytical results are expressed in such a way that the stress intensity factor may be determined from known conditions of specimen geometry and loading.

As the crack length decreased, the results obtained by the collocation procedure approached those derived from a closed solution for an edge crack in a semi-infinite plate. Over a range of ratios of crack length to specimen width between 0.15 and 0.40 the collocation solution yielded results in very good agreement with those derived from experimental compliance measurements.

## INTRODUCTION

A method for calculating the stress distribution in a test specimen containing a single-edge crack (sharp notch) and subjected to a uniform tensile load is described herein. The results are particularly useful in determining stress intensity factors $K$ for given conditions of load and geometry and therefore permit the use of the single-edge-notch specimen in fracture toughness testing.

The ASTM Special Committee on Fracture Testing of High Strength Metallic Materials issued a series of reports describing recent developments in fracture toughness testing (refs. 1 to 4). It has been shown that the magnitude of the elastic stress field in the immediate vicinity of a crack but beyond the crack tip plastic zone may be characterized by a single parameter $K$, the stress intensity factor (refs. 1 and 5). For any given material a characteristic value $K_c$ of the stress intensity factor is assumed to exist that corresponds to the onset of rapid fracture. Like other mechanical properties, $K_c$ is dependent on the strain rate, the temperature, and the testing direction. In the case of sheet and plate materials it is also dependent on the thickness. The value of $K_c$ may be determined from tests on specimens containing sharp notches or cracks, provided that suitable expressions are available that give the stress intensity factor in terms of the specimen geometry and applied loads at fracture instability.

Approximate solutions for $K$ exist in closed form for a number of specimen designs symmetrically notched with respect to the tensile load axis (refs. 5 to 9). The single-edge-notch tension specimen appears to be more efficient, however, than symmetrically notched specimens with respect both to the material and to the loading capacity required (ref. 10). For this reason, the single-edge-notch specimen may be of considerable importance in the determination of $K_c$ for plane strain crack propagation where relatively large cross sections are an inherent requirement of the test. Recent, very careful experimental compliance measurements on the single-edge-notch specimen (ref. 10) provide values of strain energy release rates as a function of crack length from which values of $K$ may be derived. An analytical solution is desirable, however, as an independent check on the experimental procedure; it also has the advantage that the influence of certain geometrical parameters, such as the ratio of height to width $V/W$, may be rapidly determined without resort to tedious experimental measurements. Furthermore, the method of obtaining an analytical solution is applicable to all combinations of bending and tension applied to a single-edge-notch specimen.

An analytical solution to the stress distribution in the single-edge-notch tension specimen is obtained herein by a boundary value collocation procedure applied to the Williams stress function (ref. 11), which is known to satisfy the boundary conditions along an edge crack. The results are in a form that permits expression of stress intensity factors in terms of the measured quantities of load and specimen dimensions. In addition, the influence of the end effect on the stress intensity factor is determined. The end effect derives from the finite distance between the crack and the uniformly loaded boundary, expressed as $V/W$. A comparison is made between the present analytical results and a closed solution obtained by Wigglesworth (ref. 12) for an edge crack in a semi-infinite plate. Finally, the collocation solution in terms of the stress intensity factor is compared with experimental results obtained by other investigators for this specimen with strain energy release rate (compliance measurement) experiments.

## SYMBOLS

$a$ crack length in single-edge-notch specimen, in.

$d_n$ coefficients of Williams stress function

$E$ Young's modulus, psi

$\mathcal{G}$ strain energy release rate with crack extension; or crack extension force, in.-lb/sq in.

| | | |
|---|---|---|
| $K$ | stress intensity factor of elastic stress field in vicinity of border of crack, psi $\sqrt{\text{in.}}$ |
| $P$ | load per unit thickness, lb/in. |
| $r,\theta$ | angular position coordinates referred to crack tip |
| $V$ | distance (height) between crack plane and location of uniform stress, in. |
| $W$ | specimen width, in. |
| $x,y$ | coordinate axes with origin at crack tip, parallel and perpendicular, respectively, to crack plane |
| $\sigma_0$ | uniform tensile stress applied to specimen, psi |
| $\sigma_x, \sigma_y, \tau_{xy}$ | stress in $x$- and $y$-directions, psi |
| $\chi$ | stress function |

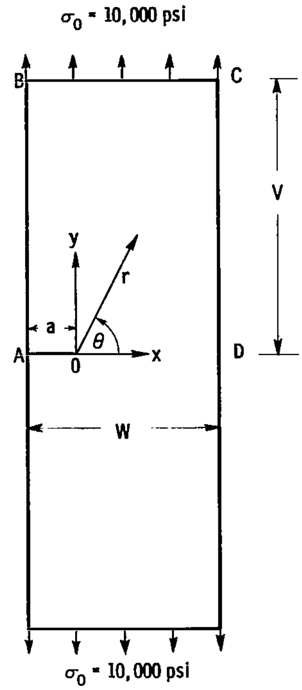

Figure 1. - Specimen geometry and loading assumed for collocation solution.

## METHOD

The method of analysis consists in finding a stress function $\chi$ satisfying the biharmonic equation $\nabla^4\chi = 0$ and the boundary conditions at a finite number of stations along the boundaries of the single-edge-notch specimen shown in figure 1. For the present purposes use is made of the Williams stress function (ref. 11) with the correction of a typographical error in that reference:

$$\chi(r,\theta) = \sum_{n=1,2,\ldots}^{\infty} \left\{(-1)^{n-1} d_{2n-1} r^{n+(1/2)} \left[- \cos\left(n - \frac{3}{2}\right)\theta + \frac{2n - 3}{2n + 1} \cos\left(n + \frac{1}{2}\right)\theta\right] + (-1)^{n} d_{2n} r^{n+1} \left[- \cos(n - 1)\theta + \cos(n + 1)\theta\right]\right\} \tag{1}$$

Because of symmetry (fig. 1) only even terms of the stress function are considered. The stresses in terms of $\chi$ obtained by partial differentiation are as follows:

$$
\left.
\begin{aligned}
\sigma_y &= \frac{\partial^2\chi}{\partial x^2} = \frac{\partial^2\chi}{\partial r^2}\cos^2\theta - 2\frac{\partial^2\chi}{\partial\theta\,\partial r}\frac{\sin\theta\cos\theta}{r} + \frac{\partial\chi}{\partial r}\frac{\sin^2\theta}{r} \\
&\qquad + 2\frac{\partial\chi}{\partial\theta}\frac{\sin\theta\cos\theta}{r^2} + \frac{\partial^2\chi}{\partial\theta^2}\frac{\sin^2\theta}{r^2} \\
\sigma_x &= \frac{\partial^2\chi}{\partial y^2} = \frac{\partial^2\chi}{\partial r^2}\sin^2\theta + 2\frac{\partial^2\chi}{\partial\theta\,\partial r}\frac{\sin\theta\cos\theta}{r} + \frac{\partial\chi}{\partial r}\frac{\cos^2\theta}{r} \\
&\qquad - 2\frac{\partial\chi}{\partial\theta}\frac{\sin\theta\cos\theta}{r^2} + \frac{\partial^2\chi}{\partial\theta^2}\frac{\cos^2\theta}{r^2} \\
-\tau_{xy} &= \frac{\partial^2\chi}{\partial x\,\partial y} = \sin\theta\cos\theta\frac{\partial^2\chi}{\partial r^2} + \frac{\cos 2\theta}{r}\frac{\partial^2\chi}{\partial r\,\partial\theta} - \frac{\sin\theta\cos\theta}{r^2}\frac{\partial^2\chi}{\partial r\,\partial\theta} \\
&\qquad - \frac{\sin\theta\cos\theta}{r}\frac{\partial\chi}{\partial r} - \frac{\cos 2\theta}{r^2}\frac{\partial\chi}{\partial\theta}
\end{aligned}
\right\} \qquad (2)
$$

The Williams stress function is an Airy stress function, which, besides satisfying the biharmonic equation, also satisfies the boundary conditions along the crack surface, namely, that the normal and shearing stresses be zero. Thus, when $\theta = \pm\pi$, equations (1) and (2) give $\sigma_y = 0$, $\tau_{xy} = 0$. The remaining boundary requirements on the stress function for the specimen having the geometry and tractions shown in figure 1 are as follows:

$$
\left.
\begin{aligned}
&\text{Along boundary } A \to B: \\
&\qquad \chi = 0, \quad \frac{\partial\chi}{\partial x} = 0 \\
&\text{Along boundary } B \to C: \\
&\qquad \chi = \sigma_o\left(\frac{x^2}{2} + ax + \frac{a^2}{2}\right), \quad \frac{\partial\chi}{\partial y} = 0 \\
&\text{Along boundary } C \to D: \\
&\qquad \chi = \frac{\sigma_o W^2}{2}, \quad \frac{\partial\chi}{\partial x} = \sigma_o W
\end{aligned}
\right\} \qquad (3)
$$

Because of symmetry with respect to the crack plane (fig. 1) only half the specimen need be considered.

For the purpose of determining the stress intensity factor as defined in reference 13, which characterizes the stress distribution in the immediate neighborhood of the crack tip ($r \to 0$), only the first coefficient $d_1$ of the

Williams stress function is necessary, since this term is dominant. As shown later, $d_1$ is proportional to the stress intensity factor K. Values of $d_1$ as well as of the other coefficients are obtained by satisfying the boundary conditions (eq. (3)) at a finite number of stations equally spaced along a given boundary for a specimen with the geometry shown in figure 1 that is subjected to a uniform stress of 10,000 psi acting at a distance V from the crack plane. Computations were made for several ratios of crack length to specimen width a/W between 0.04 and 0.5 and for values of V/W ranging from 0.5 to 1.5.

The collocation procedure requires a matrix solution of twice as many equations as the number of boundary stations selected for each combination of the independent variables. This problem was programed for a digital computer with the use of double precision arithmetic (16 significant figures).

In this solution, the number of boundary stations is increased until the first matrix coefficient $d_1$ converges to a sufficiently stable value. Figure 2, for example, shows the first matrix coefficient as a function of the number of boundary stations for configurations with several V/W ratios at a value of a/W of 1/3. The variation in the first matrix coefficient is not more than ±1 percent when the number of boundary stations is increased from 11 to 23.

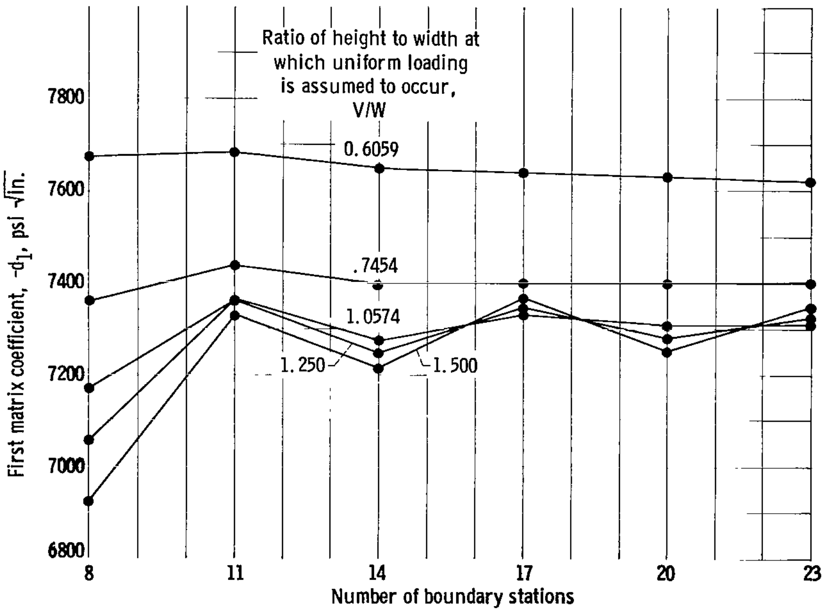


Figure 2. - First matrix coefficient as function of number of boundary stations. Tensile stress applied to specimen, 10,000 psi; specimen width, 1 inch; ratio of crack length to specimen width, 1/3.

The stress-function values at 50 stations along the boundary were computed for several geometries with all $d_n$ coefficients. These values were in good agreement with the prescribed values. The stress-function derivative normal to the boundary, however, showed perturbations near the corners of the specimen. The effect of this variation in terms of stress distribution throughout the specimen could be determined only by additional computation. This additional effort did not appear justified since the first matrix coefficient was, for practical purposes, insensitive to these perturbations.

## RESULTS

### Stress Intensity Factors

The stress intensity factor K may be derived in terms of the first coefficient of the Williams stress function. The expression for the stress in the y-direction in the immediate vicinity of the crack tip is obtained from the dominant term as follows:

$$\sigma_y = \frac{-d_1}{\sqrt{r}} \cos \frac{\theta}{2} \left(1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2}\right) \tag{4}$$

The expression for $\sigma_y$ given in reference 1, based on the Westergaard crack stress analysis, is as follows:

$$\sigma_y = \frac{K}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2}\right) \tag{5}$$

Thus

$$K = -d_1 \sqrt{2\pi} \tag{6}$$

As shown in figure 2, small oscillations sometimes occur in the first matrix coefficient. For this reason stress intensity factors were computed with values of $d_1$ averaged from 11 to 23 boundary points.

For purposes of fracture toughness testing the results of the collocation procedure are conveniently expressed in the form of a dimensionless parameter involving K and the measured quantities as a function of a/W (ref. 10). Thus,

$$\frac{K^2}{\sigma_o^2 W} = \frac{K^2 W}{P^2} \tag{7}$$

where P is the load per unit thickness. As discussed later, this form is useful for a comparison of the analytical results with experimental compliance calibration data. A curve derived from equation (3) relating $K^2W/P^2$ to a/W is given in figure 3. This curve applies to any V/W value greater than about 0.8 (see fig. 4).

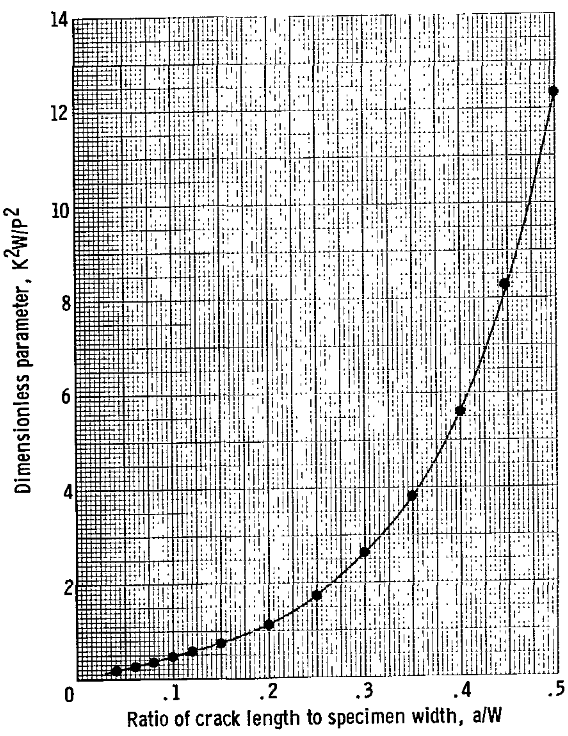


Figure 3. - Collocation results of a plot of dimensionless parameter against ratio of crack length to specimen width in single-edge-notch specimen.

## Influence of End Effects

As an aid to optimizing the specimen design, calculations were made to show how close the assumed position of the uniformly loaded boundary could be to the crack plane without affecting the stress intensity factor. For a given value of a/W the position, of course, is a function of the specimen width, and expressing the first matrix coefficient as a function of the ratio of height to width (see fig. 1, p. 3) is convenient for various values of a/W. According to figure 4, the first matrix coefficient at a/W ratios between 0.15 and 0.5 is essentially constant for ratios of height to width V/W greater than about 0.8.

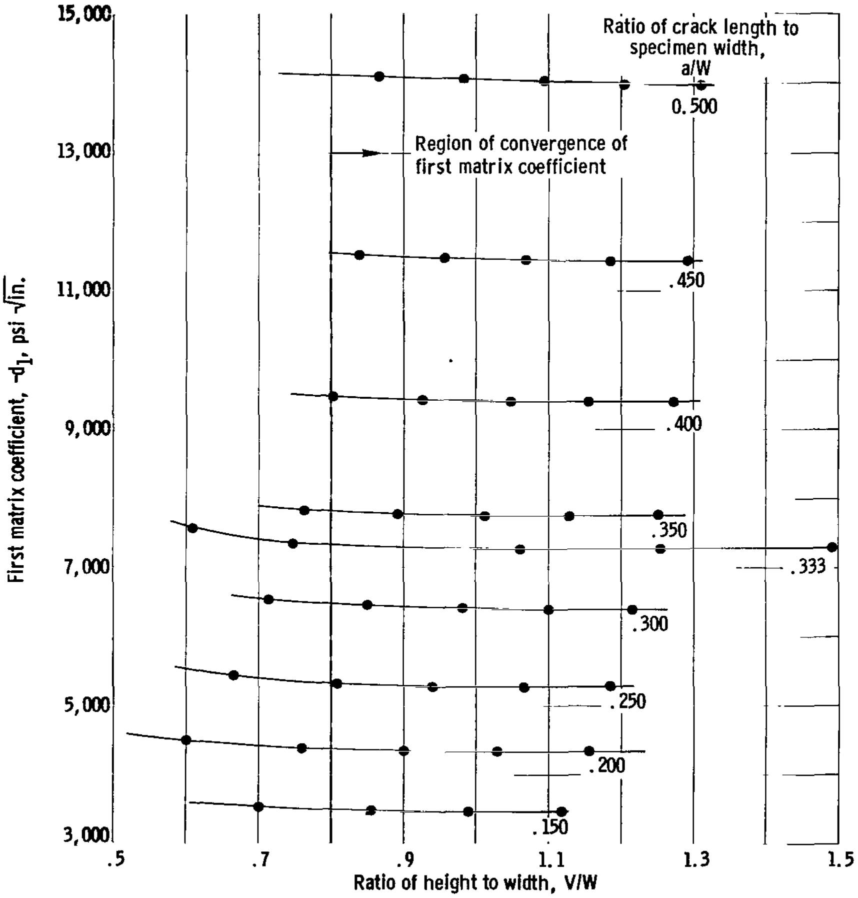


Figure 4. - First matrix coefficient as function of ratio of height to width for various ratios of crack length to specimen width. Tensile stress applied to specimen, 10,000 psi; specimen width, 1 inch.

In a practical test specimen the means by which load is applied introduces nonuniformly stressed regions at the ends of the specimen. The actual specimen length must therefore exceed the minimum length determined from figure 4. In the case of a pin-loaded specimen, for example, the optimum ratio of total length to width is about 4 (ref. 10).

## DISCUSSION OF RESULTS

### Comparison with Wigglesworth Solution

A check on the validity of the present solution can be obtained by comparison with that reported by Wigglesworth (ref. 12) for an edge crack in a semi-infinite plate. These two solutions should converge as the crack length tends to zero. In the immediate vicinity of the crack the first term of the Wigglesworth solution predominates, and $\sigma_y$ may be expressed in terms of the coordinate system shown in figure 1 (p. 3) as follows:

$$\sigma_y = 0.793\ \sigma_o \sqrt{\frac{a}{r}} \cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) \tag{8}$$

This equation may be compared with the corresponding expression obtained by the present method:

$$\sigma_y = \frac{-d_1}{\sqrt{r}} \cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) \tag{4}$$

where the first matrix coefficient $d_1$ depends on $\sigma_o$ and the specimen dimensions. The respective values of $\sigma_y$ in any direction near the crack tip may be compared by considering a single-edge-notch specimen of unit width. For the same uniform stress in both cases, the ratio of $\sigma_y$ obtained from the Wigglesworth solution (eq. (8)) to that computed from equation (4) should ap-

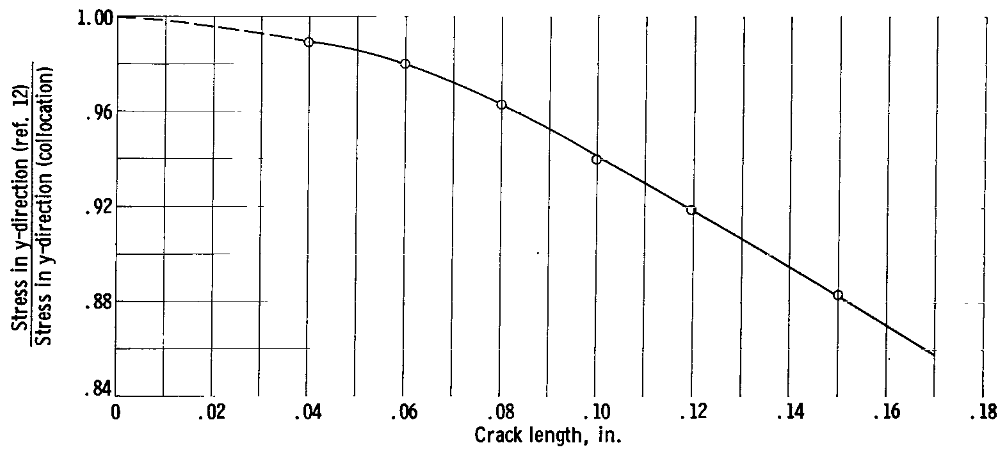


Figure 5. - Comparison of stress ratios in immediate vicinity of crack tip obtained by collocation solution and solution of reference 12.

proach 1 as the crack length decreases. The results presented in figure 5 are in accordance with this behavior.

### Comparison with Experimental Results

Two experimental compliance calibrations of single-edge-notch specimens loaded in tension are available for comparison, the earlier by Sullivan (ref. 14) and a more recent one by Srawley, Jones, and Gross (ref. 10). The design of the specimen of reference 14 was loaded through pins separated by a distance less than twice the width, which introduced large end effects that are not accounted for in the analytical solution. The specimen used in reference 10 was of sufficient length that end effects were negligible, since the compliance measurements were made over a gage length of 8 inches on a specimen 3 inches wide that was loaded through pins 10 inches apart. The sufficiency of this gage length was established by preliminary experiments. For this reason better agreement with the analytical results is to be expected from the experiments of reference 10 than from those of reference 14. Furthermore, as discussed in reference 10, those data are expected to be more precise than the data of reference 14 because of differences in specimen size and measurement techniques.

The experimental compliance procedure gives results in terms of the strain energy release rate $\mathscr{G}$. The correct procedure for converting these values of $\mathscr{G}$ to stress intensity factors is not yet completely settled (see ref. 10). For the purposes of comparing the analytical with the experimental results the most reasonable procedure appears to be a conversion on the plane stress basis. Thus,

$$K^2 = E\mathscr{G}$$

or in terms of experimentally measured quantities,

$$\frac{K^2W}{P^2} = \frac{E\mathscr{G}W}{P^2} \tag{9}$$

where $\mathscr{G}$ is determined by experimental compliance procedures and $P$ is the load per unit specimen thickness.

| Ratio of crack length to specimen width, a/W | Dimensionless parameter, $K^2W/P^2$ | | |
|---|---|---|---|
| | Experimental results | | Collocation results |
| | Ref. 10 | Ref. 14 | |
| 0.05 | 0.314 | 0.35 | 0.204 |
| .10 | .556 | .65 | .445 |
| .15 | .816 | 1.00 | .758 |
| .20 | 1.180 | 1.40 | 1.180 |
| .25 | 1.735 | 1.97 | 1.768 |
| .30 | 2.571 | 2.80 | 2.603 |
| .35 | 3.775 | 4.20 | 3.813 |
| .40 | 5.436 | 6.18 | 5.596 |
| .45 | 7.641 | 8.90 | 8.276 |
| .50 | 10.477 | 12.50 | 12.399 |

The comparison between experimental and analytical results is shown in the table. As might be expected from the foregoing discussion of the compliance calibration experiments, the results given in reference 14 are consistently higher than either those obtained by the present collocation solution or those reported in reference 10. In contrast, very good agreement between the analytical solution and the data of reference 10 is noted for values of $a/W$ between 0.15 and 0.40. The differences between these two sets of results at the lower values of $a/W$ are probably associated with uncertainties in the lower range of the experimental data. For the $a/W$ values above 0.4, differences due to bending of the experimental compliance specimen, which are not taken into account by the analytical solution, become important. This bending decreases the eccentricity of loading with respect to the uncracked section, and the compliance for a given slot length is therefore slightly less than if no bending took place.

## SUMMARY OF RESULTS

The results of an analytical investigation of the stress intensity factors for a single-edge-notch tension specimen obtained by a boundary value collocation procedure applied to the Williams stress function are as follows:

1. The values of the stress intensity factor were independent of the distance between the uniformly loaded cross section and the notch plane provided that this distance was greater than 80 percent of the width.

2. At small ratios of crack length to specimen width the present results were in good agreement with a closed solution obtained for an edge crack in a semi-infinite plate.

3. When the analytical results were expressed in appropriate dimensionless form, very good agreement was obtained with comparable results obtained from a highly accurate experimental strain energy release rate determination.

Lewis Research Center
National Aeronautics and Space Administration
Cleveland, Ohio, May 5, 1964

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